Irreducibility properties of Keller maps

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Abstract

Bakalarski showed that a polynomial map over a field of characteristic zero is invertible, if and only if the corresponding endomorphisms maps irreducible polynomials to irreducible polynomials. Jędrzejewicz showed that a polynomial map over a field of characteristic zero is a Keller map, if and only if the corresponding endomorphism maps irreducible polynomials to square-free polynomials. We show that the latter endomorphism maps other square-free polynomials to square-free polynomials as well.

In connection with Bakalarski's result and the Jacobian Conjecture, we study irreducible properties of several types of Keller maps, to each of which the Jacobian Conjecture can be reduced. Herewith, we generalize the result of Bakalarski that the components of cubic homogeneous unipotent Keller maps are irreducible.

Furthermore, we show that the Jacobian conjecture can even be reduced to any of these types with the extra condition that each affinely linear combination of the components of the polynomial map is irreducible. This is somewhat similar to reducing the planar Jacobian conjecture to the so-called (planar) weak Jacobian conjecture by Kaliman.

Keywords. Jacobian Conjecture, Keller map, irreducible, square-free, weak Jacobian Conjecture.

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1 Introduction

Throughout this paper, we will write x for the n indeterminates $x_1, x_2, \dots, x_n$, where $n \in \mathbb{N}$. In a similar manner, we will write y for $y_1, y_2, \dots, y_n$ and z for $z_1, z_2, \dots, z_n$. Let $F = (F_1, F_2, \dots, F_m) \in K[x]^m$, where K is a field of characteristic zero. Then F corresponds to the polynomial mapping $K^n \ni v \mapsto F(v) \in K^m$. Write $\mathcal{J}F$ for the Jacobian of F with respect to x, i.e.

$$\mathcal{J}F := \mathcal{J}_x F := \begin{pmatrix} \frac{\partial}{\partial x_1}F_1 & \frac{\partial}{\partial x_2}F_1 & \cdots & \frac{\partial}{\partial x_n}F_1 \\ \frac{\partial}{\partial x_1}F_2 & \frac{\partial}{\partial x_2}F_2 & \cdots & \frac{\partial}{\partial x_n}F_2 \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x_1}F_m & \frac{\partial}{\partial x_2}F_m & \cdots & \frac{\partial}{\partial x_n}F_m \end{pmatrix}$$

Let M^{t} denote the transpose of a matrix M. For a single polynomial $f \in K[x]$, write ∇f for the gradient of f with respect to x, i.e.

$$\nabla f := \nabla_x f := (\mathcal{J}_x f)^{\mathrm{t}} = \begin{pmatrix} \frac{\partial}{\partial x_1} f \\ \frac{\partial}{\partial x_2} f \\ \vdots \\ \frac{\partial}{\partial x_n} f \end{pmatrix}$$

Additionally, write $\mathcal{H}f$ for the Hessian of f with respect to x, i.e.

$$\mathcal{H}f := \mathcal{J}_x\big(\nabla_x f\big) = \begin{pmatrix} \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} f & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} f & \cdots & \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_1} f \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} f & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} f & \cdots & \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_2} f \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_n} f & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_n} f & \cdots & \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n} f \end{pmatrix}$$

The well-known Jacobian Conjecture (JC), raised by O.H. Keller in 1939 in [Kel], states that a polynomial mapping $F : K^n \to K^n$ is invertible if its Jacobian determinant $\det \mathcal{J}F$ is a nonzero constant. This conjecture has being attacked by many people from various research fields and remains open even when $n = 2$! (Of course, a positive answer is obvious when $n = 1$.) See [BCW] and [vdE] and the references therein for a wonderful 70-years history of this famous conjecture. The condition that $\det \mathcal{J}F \in K^*$ is called the *Keller condition* and polynomial maps that satisfy this condition are called *Keller maps.*

Among the vast interesting and valid results, a relatively satisfactory result obtained by S.S.S.Wang in [Wan] in 1980 is that the JC holds for all polynomial mappings of degree 2 in all dimensions. The most powerful and surprising result is the reduction to degree 3, due to H. Bass, E. Connell and D. Wright in [BCW] in 1982 and A. Yagzhev in [Jag] in 1980, which asserts that the JC is true if the JC holds for all polynomial mappings $F = x + H$, such that H is cubic homogeneous, i.e. each component H_i of H is either zero or a cubic form.

Upon this reduction to the cubic homogeneous case, there are two subsequent reductions, but they cannot be applied both. The first one is that additionally, H_i is a third power of a linear form for each i, see [Dru1]. The second one is that $\mathcal{J}H$ or equivalently $\mathcal{J}F$ is symmetric, see [dBvdE2]. By a special case of Poincaré's lemma, this is the same as that $F = \nabla f$ and $H = \nabla h$ for certain polynomials $f, h \in K[x]$. If both H_i is a (third) power of a linear form and $H_i = \frac{\partial}{\partial x_i} h$ for each i, then $x - H$ is the inverse of $F = x + H$, see [dBvdE1] and [Dru2].

In [Kal], the author S. Kaliman shows that in order to prove the Jacobian conjecture in dimension $n = 2$, one may assume that $F_1 + c$ is irreducible for every $c \in K$. To prove the Jacobian conjecture in dimension $n \geq 3$, one may even assume that $F_i + c$ is irreducible for every $i \leq n$ and $c \in K$. This was proved in [KS, Th. 3]. We shall show that for the Jacobian conjecture for all n, one may even assume that every affinely linear combination of the components of F is irreducible. Furthermore, we combine this reduction with several other reductions of the Jacobian conjecture, including both reductions in the previous paragraph. See theorems 3.1 and 3.2.

In [Bak1, Th. 3.7], the author S. Bakalarski proves the following interesting connection between invertible polynomial maps and irreducibility: a polynomial map from K^n to K^n is invertible, if and only if the corresponding endomorphism maps irreducible polynomials to irreducible polynomials. This result is reproved in [Jęd, Th. 5.2] by P. Jędrzejewicz. In [Jęd, Th. 5.1], Jędrzejewicz proves the following counterpart of this result: a polynomial map from K^n to K^n is a Keller map, if and only if the corresponding endomorphism maps irreducible polynomials to square-free polynomials. We shall show in the next section that for Keller maps, the corresponding endomorphism even maps all square-free polynomials to square-free polynomials.

In [Bak2], Bakalarski proves that each component of F is irreducible for each i if $F = x + H$, $\det \mathcal{J}F = 1$, H is cubic homogeneous and $\mathcal{J}H$ is symmetric. Notice that F_i is the image of x_i under the corresponding endomorphism of F. We will generalize this result in i) of theorem 3.5, where we show that F_i is irreducible if $\mathcal{J}F$ is symmetric, $\det \mathcal{J}F \in K^*$ and $F_i = l + h$ such that h and l are homogeneous and $\frac{\partial}{\partial x_i} l \in K^*$. Notice that, as opposed to the result of Bakalarski, the conditions on l and h are for F_i only, and not for the F_j with $j \neq i$.

Additionally, we show in corollary 4.4 that F_i is irreducible if $\det \mathcal{J}F \in K^*$ and the set of degrees of nonzero terms of F_i is $\{0, 1, 3\}$. If we combine this result with the above-mentioned result of theorem 3.5, we can conclude that $F_i + c$ is irreducible for all $c \in K$ if $\mathcal{J}F$ is symmetric, $\det \mathcal{J}F \in K^*$, and $F = l + h$ such that h is cubic homogeneous, $\deg l = 1$, and $\frac{\partial}{\partial x_i} l \in K^*$. The latter result can also be found in i) of theorem 3.5.

At last in this introduction, we summarize some results in connection with coordinates. A polynomial $f \in K[x]$ is a *coordinate* if there exists an invertible polynomial map $F \in K[x]^n$ such that $f = F_1$. After some partial results in [vdES] and [Jel1], Z. Jelonek proved in [Jel2] that a polynomial map over K is invertible. if and only if the corresponding endomorphism maps coordinates to

coordinates. The result of [vdES, Lm. 2.3] by H. Derksen is that a polynomial map over an algebraically closed field $\bar{K}$ is a Keller map, if and only if the corresponding endomorphism maps linear coordinates to polynomials with unipotent gradients (for instance coordinates).

By observing that a coordinate can be mapped to x_1 by way of a Keller map, one can show that Derksen's result is still valid when we replace 'linear coordinates' by 'coordinates'. It is however not true in general that a polynomial map over K is invertible, if and only if the corresponding endomorphism maps *linear* coordinates to coordinates, see [MSY] (so Derksen's result is only valid when $K = \bar{K}$). But C. Cheng and A. van den Essen proved in [CvdE, Th. 1.1] that it indeed suffices to look at the images of linear coordinates in case $n = 2$. Furthermore, A. van den Essen and V. Shpilrain show in the proof of [vdES, Th. 1.2] that Keller maps F are invertible in case F_1 is a coordinate and the Jacobian conjecture holds in dimension $n - 1$.

2 Some properties of Keller maps

We start with a generalization of [Jęd, Th. 4.1] by Jędrzejewicz. To be precise, [Jęd, Th. 4.1] is the equivalence of 1) and 2) in the theorem below, for the case that g is irreducible.

Theorem 2.1. *Let K be a field of characteristic zero and $F \in K[x]^n$ be an arbitrary polynomial map. If $g \in K[x]$ is* square-free*, then the following conditions are equivalent:*

1) *g divides $\det \mathcal{J}F$,*

2) *for every* irreducible *$\tilde{g} \mid g$, there exists an* irreducible *polynomial $\tilde{w} \in K[y]$ such that $\tilde{g}^2 \mid \tilde{w}(F)$,*

3) *$g^2 \mid w(F)$ for some* square-free *polynomial $w \in K[y]$.*

Proof. Assume that $g \in K[x]$ is square-free. The equivalence of 1) and 2) follows by applying [Jęd, Th. 4.1] for all irreducible polynomials $\tilde{g} \mid g$. To prove 2) $\Longrightarrow$ 3), take for w in 3) the least common multiple of all $\tilde{w}$ appearing in 2). Then $w(F)$ in 3) is a common multiple of the $\tilde{w}(F)$ appearing in 2). Since g^2 in 3) is the least common multiple of the $\tilde{g}^2$ appearing in 2), 2) $\Longrightarrow$ 3) follows. Hence it remains to show 3) $\Longrightarrow$ 2).

So assume 3) and let $\tilde{g}$ be an arbitrary irreducible divisor of g. We have to show that there exists an irreducible $\tilde{w} \in K[y]$ such that $\tilde{g}^2 \mid \tilde{w}(F)$. Since $g^2 \mid w(F)$, we can decompose $w = w_1 w_2$, such that w_1 is irreducible and $\tilde{g} \mid w_1(F)$. If $\tilde{g}^2 \mid w_1(F)$, then we are done, so suppose that $\tilde{g}^2 \nmid w_1(F)$. Then $\tilde{g} \mid w_2(F)$. Let $\bar{F}$ be the residue classes of F modulo $\tilde{g}$, i.e. $\bar{F}_i = F_i + (\tilde{g})$ for each i. Define $r := \operatorname{trdeg}_K(\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_n)$ and assume without loss of generality that $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_r$ are algebraically independent of K. Then $r \le n-1$, because $w_1(\bar{F}) = 0$.

If $r \le n-2$, then we can follow the last paragraph in the proof of (i) $\Longrightarrow$ (ii) of [Jęd, Th. 4.1] verbatim to obtain that $\tilde{g}^2 \mid \tilde{w}(F)$ for some irreducible $\tilde{w} \in K[y]$. So assume that $r = n-1$. Notice that w_1 and w_2 are relatively prime, because w is square-free. Now we can obtain a contradiction with $r = n-1$, either by observing that the ideal (w_1, w_2) has height at least two, or by computing the resultant with respect to y_n of w_1 and w_2. □

Just like [Jęd, Th. 4.1], its immediate consequence [Jęd, Cor. 4.2] can be generalized. We do this by extending it with one line, namely property 3).

Corollary 2.2. *Let K be a field of characteristic zero and $F \in K[x]^n$ be an arbitrary polynomial map. Then the following conditions are equivalent:*

1) $\det \mathcal{J}F \in K^*$,
2) *for every* irreducible *polynomial $w \in K[y]$, the polynomial $w(F)$ is square-free,*
3) *for every* square-free *polynomial $w \in K[y]$, the polynomial $w(F)$ is square-free.*

Proof. Every assertion is equivalent to the nonexistence of an irreducible g in the respective assertion of theorem 2.1. □

We end this section with a theorem about some reducibility properties which cannot be combined with the Keller condition. We use a result of [LM] for that.

Theorem 2.3. *Let K be a field of characteristic zero and $F \in K[x]^n$ be a Keller map. Suppose that F_i is of the form $L_i H_i$, where $\deg L_i = 1$. Then for*

1) *The linear part of $L_i(0)H_i$ is divisible by $L_i - L_i(0)$ for each i,*
2) *$L(0)$ is contained in the column space of $\mathcal{J}L$,*
3) $\det \mathcal{J}L \in K^*$,
4) $\deg F = 1$,
5) *F_i is irreducible for each i,*

we have 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) $\Longrightarrow$ 4) $\Longrightarrow$ 5).

Proof. Notice that 3) $\Longrightarrow$ 2) and 4) $\Longrightarrow$ 5) are trivial. Hence it suffices to prove the following.

1) ⇒ 3) Let $*$ denotes the Hadamard product. Suppose that 1) holds. Then for each i, the linear part of $L_i(0)H_i$ is of the form $\big(L_i - L_i(0)\big)\big(c_i - H_i(0)\big)$ for some $c_i \in K$. Hence for each i, the linear part of $F_i = L_i H_i$, which is $\big(L_i - L_i(0)\big)H_i(0)$ larger than the linear part of $L_i(0)H_i$, is equal to $\big(L_i - L_i(0)\big)c_i$. Thus the linear part of F is equal to $\big(L - L(0)\big) * c$ for some $c \in K^n$. Hence $c_1 c_2 \cdots c_n \det \mathcal{J}L = (\det \mathcal{J}F)|_{x=0}$. Now 3) follows from the Keller condition on F.

2) $\Rightarrow$ 3) Suppose that 2) holds. Say that $\mathcal{J}L \cdot a = L(0)$, where $a \in K^n$. Then the constant part of $L(x-a)$ is equal to

$$L(-a) = \mathcal{J}L|_{x=-a} \cdot (-a) + L(0) = \mathcal{J}L \cdot (-a) + \mathcal{J}L \cdot a = 0$$

Hence the linear part of $F(x-a) = L(x-a) * H(x-a)$ is equal to $L(x-a) * H(-a)$. Using the Keller condition for F, $\det \mathcal{J}L \mid \det(\mathcal{J}F)|_{x=x-a} \in K^*$ follows, which is 3).

3) $\Rightarrow$ 4) Suppose that 3) holds. Then L is invertible and

$$F\big(L^{-1}(x)\big) = L\big(L^{-1}(x)\big) * H\big(L^{-1}(x)\big) = x * H\big(L^{-1}(x)\big)$$

is a Keller map as well. It follows from [LM, Prop. 6] that $\deg H = \deg H(L^{-1}x) = 0$. Hence $\deg F = 1$. $\square$

3 Irreducibility results by reduction of general Keller maps, combined with other reductions of the JC

Theorem 3.1. *Assume $F \in K[x]^n$ is a cubic Keller map without quadratic terms over a field K of characteristic zero. Then there exists a $\lambda \in K^n$ such that for*

$$G = (F - \lambda x_{n+1}^3, x_{n+1}) \tag{1}$$

$$G = (F - \lambda x_{n+1}^3, x_{n+1}, x_{n+2} + x_{n+1}^3) \tag{2}$$

and

$$\begin{aligned} G &= (F - \lambda x_{n+1}^3, x_{n+2} - 3x^{\mathrm{t}}\lambda x_{n+1}^2, x_{n+1}) \\ &= (F, 0, 0) + \nabla_{x,x_{n+1},x_{n+2}}\big(x_{n+1}x_{n+2} - x^{\mathrm{t}}\lambda x_{n+1}^3\big) \end{aligned} \tag{3}$$

every linear combination of the components of G and 1 which is reducible is already a linear combination of 1.

Furthermore, G is a cubic Keller map without quadratic terms, and F is invertible, if and only if G is invertible. Additionally, we have the following.

i) If F is linearly conjugate to a Drużkowski map, then so is G in (2).

ii) If $\mathcal{J}F$ is symmetric, then so is $\mathcal{J}_{x,x_{n+1},x_{n+2}}G$ in (3).

Proof. In corollary 5.2, we will prove the first claim (the existence of λ) for

$$G = (F - \lambda x_{n+1}^3, x_{n+1}, x_{n+2} - h) \tag{4}$$

where $h \in K[x, x_{n+1}]$ is arbitrary. This immediately gives the first claim for G in (2). To obtain the first claim for G in (1) and (3), we remove the last

component of G and interchange the last two component of G respectively in (4). Thus a $\lambda \in K^n$ as given exists.

By expansion of the determinant along the $(n+2)$-th column, if present, and subsequently along the last row, we see that

$$\begin{aligned}\det \mathcal{J}_{x,x_{n+1}} G &= \det \mathcal{J} F \text{ in (1)},\\ \det \mathcal{J}_{x,x_{n+1},x_{n+2}} G &= \det \mathcal{J} F \text{ in (2), and}\\ \det \mathcal{J}_{x,x_{n+1},x_{n+2}} G &= -\det \mathcal{J} F \text{ in (3)}.\end{aligned}$$

Hence G is a Keller map, and one can easily verify that G is a cubic Keller map without quadratic terms.

We only prove the rest of this theorem for the cases (2) and (3), since the case (1) is similar. Let $E = (x - \lambda x_{n+1}^3, x_{n+1}, x_{n+2})$.

i) Assume that G is as in (2). Then

$$G = E(F, x_{n+1}, x_{n+2})\big|_{x_{n+2}=x_{n+2}+x_{n+1}^3}$$

Consequently, F is invertible, if and only if G is invertible.

Suppose that $TF(T^{-1}x)$ is a Drużkowski map. Set

$$\tilde{T} = \left(\begin{array}{ccc|cc} & & & 0 & \\ & T & & \vdots & \lambda \\ & & & 0 & \\ \hline 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{array}\right)$$

Then $\tilde{T}G(\tilde{T}^{-1}(x, x_{n+1}, x_{n+2}))$ is a Drużkowski map as well. Hence G is linearly conjugate to a Drużkowski map in case F is.

ii) Assume that G is as in (3). Then

$$(G_1, G_2, \ldots, G_n, G_{n+2}, G_{n+1}) = E(F, x_{n+1}, x_{n+2})\big|_{x_{n+2}=x_{n+2}-3x^{\mathrm{t}}\lambda x_{n+1}^2}$$

Consequently, F is invertible, if and only if G is invertible.

Suppose that $\mathcal{J}F$ is symmetric. Then by (3), $\mathcal{J}G$ is symmetric as well, which completes the proof. $\square$

Theorem 3.2. *Assume that $F \in K[x]^n$ is a Keller map over a field K of characteristic zero and let $d \geq 2$ be an integer. Then for*

$$G = (F - y^{*d}, y) \tag{5}$$

$$G = (F - y^{*d}, y, z + y^{*d}) \tag{6}$$

and

$$\begin{aligned}G &= (F - y^{*d}, z - dx^{\mathrm{t}} y^{*(d-1)}, y)\\ &= (F, 0, \ldots, 0, 0, \ldots, 0) + \nabla_{x,y,z}\big(y^{\mathrm{t}} z - x^{\mathrm{t}} y^{*d}\big)\end{aligned} \tag{7}$$

every linear combination of the components of G and 1 which is reducible is already a linear combination of 1.

Furthermore, G is a Keller map and F is invertible, if and only if G is invertible, and we have the following.

i) If $F - x$ is linearly conjugate to a power linear map of degree d, then so is $G - (x, y, z)$ in (6).

ii) If $\mathcal{J}F$ is symmetric, then so is $\mathcal{J}_{x,y,z}G$ in (7).

Proof. In corollary 4.9, we will prove the first claim for

$$G = (F - y^{*d}, y, z - H) \tag{8}$$

where $H \in K[x, y]^n$ is arbitrary. The proof of this theorem is a multidimensional variation of the proof of theorem 3.1, where corollary 4.9 is used by way of (8) instead of corollary 5.2 by way of (4). □

Theorem 3.3. *Assume that K is a field of characteristic zero with algebraic closure $\bar{K}$. Let $F \in K[x]^n$ be a Keller map of degree d. Take $i \in \{1, 2, \ldots, n, n+1\}$ and fix $\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_{n+1} \in \bar{K}$. If $\mu_j \neq 0$ for some j with $i \neq j \leq n$, then*

$$f := \mu_1 F_1 + \mu_2 F_2 + \cdots + \mu_n F_n + \mu_{n+1}$$

is reducible over $\bar{K}$ for at most $d^2 - 1$ values of $\mu_i \in \bar{K}$.

Proof. Assume without loss of generality that $j = 1$. Then $(f, F_2, \ldots, F_n)$ is a Keller map as well. By expansion of the Jacobian determinant along the first row, we see that $\gcd\{\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \ldots, \frac{\partial}{\partial x_n} f\} \in K^* \subseteq \bar{K}^*$. If $i \leq n$, then expansion of the Jacobian determinant along the first and the i-th row gives $\gcd\{\det \mathcal{J}_{x_k,x_l}(f, F_i) \mid 1 \leq k < l \leq n\} \in K^* \subseteq \bar{K}^*$.

We will prove the case $i = n + 1$ in 2) $\Longrightarrow$ 4) of lemma 4.5, where 2) is the opposite of the claim to be proved and 4) is the opposite of $\gcd\{\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \ldots, \frac{\partial}{\partial x_n} f\} \in \bar{K}^*$. We will prove the case $i \leq n$ in 1) $\Longrightarrow$ 3) of lemma 4.6, where 1) is the opposite of the claim to be proved and 3) is the opposite of $\gcd\{\det \mathcal{J}_{x_k,x_l}(f, F_i) \mid 1 \leq k < l \leq n\} \in \bar{K}^*$. □

Corollary 3.4. *Assume that K is a field of characteristic zero and $F \in K[x]^n$ is a Keller map. Then there exists a $\lambda \in K^n$ and a $T \in \mathrm{GL}_n(K)$ such that the components of both $F + \lambda$ and $T^{-1}F(Tx)$ are all irreducible over $\bar{K}$.*

Proof. Notice that $\#K \geq d^2$ because $\operatorname{chr} K = 0$. The first claim follows from the case $i = n + 1$ of theorem 3.3 and the second claim follows from the case $i \leq n$ of theorem 3.3 with $\mu_{n+1} = 0$. □

In [KS, Th. 3], the authors prove additional properties for the $T \in \mathrm{GL}_n(K)$ when $n \geq 3$, namely that there exists a $T \in \mathrm{GL}_n(K)$ such that for every $\lambda \in K^n$, every component of $T^{-1}F(Tx) + \lambda$ is irreducible over $\bar{K}$.

i) of theorem 3.5 below is a generalization of [Bak2, Th. 2.2] by Bakalarski. In ii) of theorem 3.5, (9) with $u = \mathrm{i}$ and $u' = -\mathrm{i}$ corresponds to the gradient

reduction of the JC in [dBvdE2], where $F - x$ is (cubic) homogeneous. Taking $u = 1$ and $u' = -1$ in (9) gives a gradient reduction of the JC that does not require imaginary units, which is essentially the gradient reduction of the JC in [Dru2].

Theorem 3.5. *Assume that G is a Keller map with a symmetric Jacobian over a field K of characteristic zero. Write $G_i^{(1)}$ for the linear part of G_i.*

i) If $G \in K[x]^n$, then G_i $(+\,c)$ is irreducible (for all $c \in K$) in case $\frac{\partial}{\partial x_i} G_i^{(1)} \neq 0$ and $G_i - G_i^{(1)}$ is (cubic) homogeneous.

ii) If $G \in K[x,y]^{2n}$, then G_i is irreducible for all i in case G is of the form

$$G = \nabla_{x,y}\big(f(x+uy) + (x+u'y)^{\mathrm{t}} F(x+uy)\big) \tag{9}$$

where $u, u' \in K$ such that $u \neq 0$, $f \in K[x]$ and $F_i \in K[x]$ for all $i \le n$. Furthermore, $u \neq u'$ and F is a Keller map in case G is of the form (9), and additionally F is invertible, if and only if G is.

Proof.

i) Suppose that $G \in K[x]^n$ and $c \in K$ such that $G_i + c$ is reducible, $\frac{\partial}{\partial x_i} G_i^{(1)} \neq 0$ and $G_i - G_i^{(1)}$ is homogeneous of degree d. Then $d \ge 2$. In order to prove i) both with and without the parenthesized parts, it suffices to obtain a contradiction in case either $d \le 3$ or $c = 0$. We shall derive a contradiction by showing that $\deg G_i = 1$ in both cases. Since G is a Keller map, we see by expansion of the Jacobian determinant of G along the i-th row that $\mathcal{J} G_i$ is unimodular.

In 1) $\Longrightarrow$ 3) of corollary 4.4, we will show that $(G_i^{(1)})^2 \mid G_i - G_i^{(1)}$ in case $G_i - G_i^{(1)}$ is homogeneous of degree $d \ge 2$ and $\mathcal{J} G_i$ is unimodular, and either $d \le 3$ or $c = 0$. In corollary 6.2, we will show that $\deg G_i = 1$ in case $\frac{\partial}{\partial x_i} G_i^{(1)} \neq 0$, $(G_i^{(1)})^2 \mid G_i - G_i^{(1)}$ and G is a Keller map with a symmetric Jacobian. Hence $\deg G_i = 1$.

ii) Suppose that $G \in K[x,y]^{2n}$. Notice first that the right hand side of (9) is equal to $\nabla_{x,y}\big((f + y^{\mathrm{t}} F)|_{(x,y)=(x+uy,x+u'y)}\big)$. Hence the chain rule for $\nabla_{x,y} = \mathcal{J}_{x,y}^{\mathrm{t}}$ tells us that

$$G = \begin{pmatrix} I_n & I_n \\ uI_n & u'I_n \end{pmatrix} \big(\nabla_{x,y}(f + y^{\mathrm{t}} F)\big)\big|_{(x,y)=(x+uy,x+u'y)}$$

In lemma 6.5, we will show that F is invertible, if and only if G is, and that $\det \mathcal{H}_{x,y}(f + y^{\mathrm{t}} F) = (-1)^n (\det \mathcal{J} F)^2$. Since G is a Keller map, we see that $u \neq u'$ and that F is a Keller map.

Again in lemma 6.5, we will show that $\mu^{\mathrm{t}} \nabla_{x,y}(f + y^{\mathrm{t}} F)$ is irreducible for all $\mu \in K^{2n}$ such that $\mu_i \neq 0$ for some $i \le n$, in case F is a Keller map. Thus G_i is irreducible for all $i \le n$. Since $u \neq 0$, G_i is irreducible for all $i > n$ as well. $\square$

The proposition below shows that $F_i - c$ is irreducible for all $c \in K$ in case F is a Keller map and $F_i - x_i$ is a power of a linear form.

Proposition 3.6. *Assume that K is a field of characteristic zero and $f \in K[x_i]$ such that $\deg f \neq 1$ and $h := f - x_i$ is a polynomial in a linear form. If $h \in K[x_i]$, then f is not a component of a Keller map. If $h \notin K[x_i]$, then f is a tame coordinate.*

Proof. If $h \in K[x_i]$, then by expansion along the Jacobian row corresponding to f, we see that f cannot be a component of a Keller map. So assume $h \notin K[x_i]$. Then there exist a $T \in \mathrm{GL}_n(K)$ such that $T_1 x = x_i$ and $h \in K[T_2 x]$, say that $h = p(T_2 x)$. It follows that f is the first component of the composition of the elementary invertible map $(x_1 + p(x_2), x_2, x_3, \ldots, x_n)$ and T. □

Below is another proposition that can be used to show that $F_i - c$ is irreducible for all $c \in K$ in some cases.

Proposition 3.7. *Assume that K is a field of characteristic zero and $f, h \in K[x]$ such that $\deg(f - h) = 1$. If there exists a vector $v \in K^n$ such that $\mathcal{J}h \cdot v = 0 \neq \mathcal{J}f \cdot v$, then f is a tame coordinate.*

Proof. Since $v \neq 0$, there exists a $T \in \mathrm{GL}_n(K)$ such that $v = Te_1$, where e_1 is the first standard basis unit vector and hence Te_1 is the first column of T. Consequently,

$$\mathcal{J}h(Tx) \cdot e_1 = (\mathcal{J}h)|_{x=Tx} \cdot v = 0 \neq (\mathcal{J}f)|_{x=Tx} \cdot v = \mathcal{J}f(Tx) \cdot e_1$$

It follows that $h(Tx) \in K[x_2, x_3, \ldots, x_n]$ and $f(Tx) \notin K[x_2, x_3, \ldots, x_n]$. Since $\deg(f(Tx) - h(Tx)) = 1$, we see that $f(Tx) - h(Tx) - cx_1 \in K[x_2, x_3, \ldots, x_n]$ for some $c \in K^*$. Hence $f(Tx) - cx_1 \in K[x_2, x_3, \ldots, x_n]$ and $E := (c^{-1} f(Tx), x_2, x_3, \ldots, x_n)$ is a elementary invertible polynomial map. Since f is the first component of $cE(T^{-1}x)$, we see that f is a tame coordinate. □

4 Irreducibility lemmas for polynomials with unimodular partial derivatives

Lemma 4.1. *Let $\bar{K}$ be an algebraically closed field and $f \in \bar{K}[x]$ such that $\deg f = d \geq 2$. Suppose that f has only terms of degree $0, 1, d$, and $x_1 - c \mid f$ for some $c \in K^*$. If f is nonsingular, then*

$$f = c'(x_1^d - c^d) + (c'' - c' d c^{d-1})(x_1 - c)$$

for some $c', c'' \in \bar{K}^*$.

Proof. Since f is nonsingular, we have $f = (x_1 - c)\big(g \cdot (x_1 - c) + c''\big)$ for some $g \in \bar{K}[x]$ and a $c'' \in \bar{K}^*$. Hence $f - c''(x_1 - c) = (x_1 - c)^2 g$ and it suffices to show that

$$(x_1 - c)^2 g = c'(x_1^d - c^d) - c' d c^{d-1}(x_1 - c) \tag{10}$$

for some $c' \in \bar{K}^*$. Let c' be the coefficient of x_1^d in f. Then the left hand side of (10), which is the difference of two polynomials with terms of degrees $0, 1, d$ only, is of the form $c'x_1^d + b'x_1 + a'$ for some $a', b' \in \bar{K}$. Since c' is already given, there is only one polynomial of the form $c'x_1^d + b'x_1 + a'$, with $a', b' \in \bar{K}$, which is divisible by $(x_1 - c)^2$. Hence it suffices to show that the right hand side of (10) is divisible by $(x_1 - c)^2$. If we divide the right hand side of (10) by $c'(x_1 - c)$, then we get

$$(x_1^{d-1} + cx_1^{d-2} + \cdots + c^{d-1}) - dc^{d-1} = \sum_{i=0}^{d-1} c^{d-1-i}(x_1^i - c^i)$$

which is again divisible by $x_1 - c$. So the right hand side of (10) is divisible by $x_1 - c$ twice. □

Lemma 4.2. *Let K be a field and $f \in K[x]$ be nonzero. Assume that $f = gh$ is a polynomial decomposition, such that $h(0) \neq 0$. Take $g^* \in K[x]$ and write $f^{(i)}$ for the homogeneous part of degree i of f.*

If $g^* \mid f^{(0)}, f^{(1)}, \ldots, f^{(\deg g)}$, then $g^* \mid g$.

Proof. Write $g^{(i)}$ for the homogeneous part of degree i of g. Notice that $g^* \mid f^{(0)} = g^{(0)}h(0) \mid g^{(0)}$. Suppose that $g^* \mid g^{(0)}, g^{(1)}, \ldots, g^{(i)}$ for some $i < \deg g$. Since $g^* \mid f^{(i+1)}$, we obtain by expressing $f^{(i+1)}$ in the homogeneous parts of g and h that $g^* \mid g^{(i+1)}h(0) \mid g^{(i+1)}$. By induction on i, $g^* \mid g^{(0)}, g^{(1)}, \ldots, g^{(\deg g)}$, so $g^* \mid g^{(0)} + g^{(1)} + \cdots + g^{(\deg g)} = g$. □

Corollary 4.3. *Let K be a field and assume $f \in K[x]$ such that $f - ax_1$ is homogeneous of degree $d \geq 2$ for some nonzero $a \in K$. If f is reducible, then $x_1 \mid f$.*

Proof. Suppose that f is reducible. Then we can decompose $f = gh$ such that $h(0) \neq 0$ and $\deg g \leq d - 1$. From lemma 4.2 with $g^* = x_1$, we obtain that $x_1 \mid g \mid f$. □

Corollary 4.4. *Let K be a field and assume that $f \in K[x]$ has terms of degree $0, 1, d$ only, where $d \geq 2$, say that $f = f^{(0)} + f^{(1)} + f^{(d)}$ such that $f^{(i)}$ is homogeneous of degree i or zero. Suppose that f is reducible and $\mathcal{J}f$ is unimodular. Then for*

1) $d \leq 3$ or $f^{(0)} = 0$,

2) f has a divisor of degree 1,

3) $f^{(0)} = 0$, $f^{(1)} \mid f$ and $(f^{(1)})^2 \mid f^{(d)}$,

we have 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3).

Proof. Since $\mathcal{J}f$ is unimodular, we have $f^{(1)} \neq 0$ and f is nonsingular over the algebraic closure $\bar{K}$ of K.

The case $d \leq 3$ of 1) $\Longrightarrow$ 2) follows directly from the supposition that f is reducible and the case $f^{(0)} = 0$ of 1) $\Longrightarrow$ 2) follows from corollary 4.3, because we may assume without loss of generality that $f^{(1)} = x_1$.

In order to prove 2) $\Longrightarrow$ 3), suppose that f has a divisor of degree 1. Without loss of generality, we may assume that $x_1 - c \mid f$ for some $c \in K$. Notice that $\deg f \geq 2$ because f is reducible. Hence $(\frac{\partial}{\partial x_1} f \ 0 \ \cdots \ 0)$ is not unimodular. If $c \neq 0$, then lemma 4.1 tells us that $\mathcal{J} f = (\frac{\partial}{\partial x_1} f \ 0 \ \cdots \ 0)$, which contradicts, that $\mathcal{J} f$ is unimodular. So $c = 0$ and $x_1 \mid f$. By the nonsingularity of f over $\bar{K}$, we obtain that $f = c' x_1 (g x_1 + 1)$ for some $c' \in K^*$ and a $g \in K[x]$. This gives 3). $\square$

Lemma 4.5. *Let $\bar{K}$ be an algebraically closed field of characteristic zero and assume that $f \in \bar{K}[x]$ has degree d, and $g \in \bar{K}[y] \setminus \bar{K}$. Then for*

1) *$f - g$ is reducible,*

2) *$f - c$ is reducible for at least d^2 values of $c \in \bar{K}$,*

3) *$f \in \bar{K}[p^2, p^3]$ for some $p \in \bar{K}[x]$,*

4) *$\gcd\left\{\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \ldots, \frac{\partial}{\partial x_n} f\right\} \notin \bar{K}^*$,*

we have 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) $\Longrightarrow$ 4).

Proof. 2) $\Longrightarrow$ 3) follows immediately from Corollary 3 of [Sch, Th. 37]. If $f \in \bar{K}$, then $\mathcal{J} f = 0$ and 4) holds. If $f \notin \bar{K}$ and 3) holds, then $0 < \deg p < \deg f$ and 4) is satisfied because $\frac{\partial f}{\partial p} \notin \bar{K}$ and $\frac{\partial f}{\partial p} \mid \frac{\partial f}{\partial x_i}$ for all i. This gives 3) $\Longrightarrow$ 4), so 1) $\Longrightarrow$ 2) remains to be proved. If $f \in \bar{K}$, then 2) is trivially satisfied, so assume that $\deg_x f \geq 1$.

Suppose that we have a decomposition $f - g = h_1 \cdot h_2$ and that $h_1 \in \bar{K}[y]$. Then the leading homogeneous part with respect to x of $f - g$ is divisible by h_1. This contradicts that $f - g$ has no terms with variables of both x and y, so $\deg_x h_1 > 0$ and $\deg_x h_2 < \deg_x f$. Consequently, $\deg_x h_2 < \deg(f - c)$ and similarly $\deg_x h_1 < \deg(f - c)$. For every $c \in \bar{K}$, $g - c$ has a solution $a \in \bar{K}^n$, and $h_1|_{y=a} \cdot h_2|_{y=a}$ is a decomposition of $f - c$, because both factors have degree less than $\deg(f - c)$. This gives 1) $\Longrightarrow$ 2). $\square$

Lemma 4.6. *Let $\bar{K}$ be an algebraically closed field of characteristic zero and assume that $f, g \in \bar{K}[x]$ have degree at most d. Then for*

1) *$f - cg$ is reducible for at least d^2 values of $c \in \bar{K}$ and $\gcd\{f, g\} \in \bar{K}$,*

2) *there exists $p, q \in \bar{K}[x]$ such that $f \in \bar{K}[p, q]$ and $g \in \bar{K}[p^2, pq, q^2, p^3, p^2 q, pq^2, q^3]$,*

3) *$\gcd\{\det \mathcal{J}_{x_i, x_j}(f, g) \mid 1 \leq i < j \leq n\} \notin \bar{K}^*$.*

we have 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3).

Proof. 1) $\Longrightarrow$ 2) follows from Corollary 2 of [Sch, Th. 37], because $f + x_{n+1}g$ is irreducible in case $\gcd\{f,g\} \in \bar{K}^*$. If $\mathcal{J}f$ and $\mathcal{J}g$ are dependent over $\bar{K}$, then the formula in 3) equals zero, so assume the opposite. Then f and g are algebraically independent over $\bar{K}$. Suppose that 2) holds. Then p and q are algebraically independent over $\bar{K}$ as well, and by the chain rule

$$\mathcal{J}_{x_i,x_j}(f,g) = \mathcal{J}_{p,q}(f,g) \cdot \mathcal{J}_{x_i,x_j}(p,q)$$

Hence $\det \mathcal{J}_{p,q}(f,g)$ divides the fomula in 3). Since both entries of $\mathcal{J}_{p,q}f$ are contained in $\bar{K}[p,q]$ and both entries of $\mathcal{J}_{p,q}g$ are contained in $\bar{K}[p,q] \setminus \bar{K}$, we have $\det \mathcal{J}_{p,q}(f,g) \in \bar{K}[p,q] \setminus \bar{K}$, and 3) follows. This gives 2) $\Longrightarrow$ 3). □

Lemma 4.7. *Assume $F \in K[x]^n$ is any polynomial map over a field K of characteristic zero with algebraic closure $\bar{K}$. Let $d \geq 2$ and $\lambda \in K^n$. Then for all $h \in K[x, x_{n+1}]$, the map $G = (F - \lambda x_{n+1}^d, x_{n+1}, x_{n+2} - h)$ has the property that for all $\mu \in K^{n+3}$, either*

$$g := \mu_1 G_1 + \mu_2 G_2 + \cdots + \mu_{n+2} G_{n+2} + \mu_{n+3}$$

is irreducible, or $\mu_{n+1} = \mu_{n+2} = 0$ and

$$f := \mu_1 F_1 + \mu_2 F_2 + \cdots + \mu_n F_n + \mu_{n+3} = g$$

or we have $\mu_{n+2} = 0$ and $f \in \bar{K}[p^2, p^3]$ for some $p \in \bar{K}[x]$, in which case $\mathcal{J}f$ is not unimodular.

Proof. Assume that $h \in K[x, x_{n+1}]$ and that g is reducible. Then $\mu_{n+2} = 0$ because otherwise g would be a tame coordinate. Hence $g - f \in K[x_{n+1}]$. Since the nonzero terms of $g - f$ have degrees 1 and d only, we even have $g - f \in K[x_{n+1}] \setminus K^*$. If $f = g$, then $\mu_{n+1} = 0$ because $\mu_{n+1}x_{n+1}$ is the difference between the linear parts of g and f. If $f \neq g$, then by 1) $\Longrightarrow$ 3) of lemma 4.5 (with $g - f$ instead of g), $f \in \bar{K}[p^2, p^3]$ for some $p \in \bar{K}[x]$, and 3) $\Longrightarrow$ 4) of lemma 4.5 tells us $\mathcal{J}f$ is not unimodular over $\bar{K}$ and hence neither over K. □

Lemma 4.8. *Assume that $F \in K[x]^n$ is any polynomial map over a field K of characteristic zero with algebraic closure $\bar{K}$. Let $\Lambda \in K[y]^n$ such that $y_1, y_2, \ldots, y_n, \Lambda_1, \Lambda_2, \ldots, \Lambda_n, 1$ are linearly independent over K. Then for all $H \in K[x,y]^n$, the map*

$$G := (F - \Lambda, y, z - H)$$

has the property that for all $\mu \in K^{3n+1}$, either

$$g := \mu_1 G_1 + \mu_2 G_2 + \cdots + \mu_{3n} G_{3n} + \mu_{3n+1}$$

is irreducible, or $\mu_{2n+1} = \mu_{2n+2} = \cdots = \mu_{3n} = 0$ and for

$$f := \mu_1 F_1 + \mu_2 F_2 + \cdots + \mu_n F_n + \mu_{3n+1}$$

we have $f \in \bar{K}[p^2, p^3]$ for some $p \in \bar{K}[x]$, in which case $\mathcal{J}f$ is not unimodular.

Proof. Assume that $H \in K[x,y]^n$ and suppose that g is reducible. Then $\mu_{2n+1} = \mu_{2n+2} = \cdots = \mu_{3n} = 0$ because otherwise g would be a tame coordinate. Hence $g - f \in K[y]$. Since 1 is linearly independent over K of $y_1, y_2, \dots, y_n, \Lambda_1, \Lambda_2, \dots, \Lambda_n$, as opposed to $g - f$, we even have $g - f \in K[y] \setminus K^*$. If $f = g$, then the linear independence over K of $y_1, y_2, \dots, y_n, \Lambda_1, \Lambda_2, \dots, \Lambda_n$ tells us that $f = \mu_{3n+1} \in \bar{K}[p^2, p^3]$ for any $p \in \bar{K}[x]$ and that $\mathcal{J} f = (0^1 \; 0^2 \; \cdots \; 0^n)$ is not unimodular. The case $f \neq g$ follows in a similar manner as in the proof of lemma 4.7. $\square$

Corollary 4.9. *Assume that $F \in K[x]^n$ is a Keller map over a field K and $d \geq 2$. Then for all $H \in K[x,y]^n$, the map*

$$G := (F - y^{*d}, y, z - H)$$

has the property that

$$\mu_1 G_1 + \mu_2 G_2 + \cdots + \mu_{3n} G_{3n} + \mu_{3n+1}$$

is irreducible for all $\mu \in K^{3n+1}$ such that $\mu_i \neq 0$ for some $i \leq 3n$.

Proof. Assume that $H \in K[x,y]^n$ and suppose that $g := \mu_1 G_1 + \mu_2 G_2 + \cdots + \mu_{3n} G_{3n} + \mu_{3n+1}$ is reducible and $\mu_i \neq 0$ for some $i \leq 3n$. Let $f := \mu_1 F_1 + \mu_2 F_2 + \cdots + \mu_n F_n + \mu_{3n+1}$. By lemma 4.8, we have $i \leq 2n$. Since $\deg g \neq 1$ by reducibility of g, we can even take $i \leq n$.

Again by lemma 4.8, $\mathcal{J} f$ is not unimodular, and expansion along the i-th row of the Jacobian determinant tells us that $(F_1, \dots, F_{i-1}, f, F_{i+1}, \dots, F_n)$ is not a Keller map. This contradicts that F is a Keller map and that f is as given with $\mu_i \neq 0$, so g is irreducible in case $\mu_i \neq 0$ for some $i \leq 3n$. $\square$

5 Irreducibility results for cubic Keller maps without quadratic terms

Theorem 5.1. *Assume $F = x + H$ is a polynomial map over a field K of characteristic zero, such that $H \in K[x]^n$ is cubic homogeneous and $\mathcal{J} H$ is nilpotent. Say that besides linear combinations of 1 only, there are exactly $s \geq 1$ linear combinations of $F_1, F_2, \dots, F_n, 1$ which are reducible, when we do not count scalar multiples.*

Then $s \leq n - 4$ and there exists a $T \in \mathrm{GL}_n(K)$ such that the first s components of $T^{-1}F(Tx)$ are reducible. In particular, s is finite and the first s components of $T^{-1}F(Tx)$ are the only linear combinations of the components of $T^{-1}F(Tx)$ and 1, which are reducible and not a linear combination of 1 only.

Proof. Notice that F is a Keller map and therefore, $\mathcal{J} F_i$ is unimodular for each i. By 1) $\Longrightarrow$ 3) of corollary 4.4, all reducible linear combinations of $F_1, F_2, \dots, F_n, 1$ are already linear combinations of $F_1, F_2, \dots, F_n$. Replace F by a linear conjugation of F such that as many components of F as possible become reducible,

say that exactly t such components become reducible. Assume without loss of generality that $F_1, F_2, \dots, F_t$ are the reducible components of F.

It suffices to show that $t = s$ and $t \le n-4$. We first show the latter by distinguishing $t > n-4$ into three cases.

- $(t >)\ n - 4 \le 0$.
  Then $n \le 4$ and we have $s = t = 0$ on account of E. Hubbers result that the Jacobian conjecture holds for F, see [Hub] or [vdE, Cor. 7.1.3]. This contradicts $s \ge 1$.

- $t > (n-4 >)\ 0$ *and for each* $i \le t$*, there exists a* $j \le t$ *such that* $H_j = \lambda_j x_i x_j^2$ *for some* $\lambda_j \in K^*$.
  Notice that j as above is unique for all i, hence $i \mapsto j$ is a permutation of $\{1, 2, \dots, t\}$, say with a cycle of length $k \le t$. Then we may assume without loss of generality that

$$H_1 = \lambda_1 x_k x_1^2 \quad H_2 = \lambda_2 x_1 x_2^2 \quad H_3 = \lambda_3 x_2 x_3^2 \quad \cdots \quad H_k = \lambda_k x_{k-1} x_k^2$$

  The leading principal minor determinant of size k of $\mathcal{J}H$ equals $\big(2^k - (-1)^k\big)\lambda_1 x_1^2 \lambda_2 x_2^2 \cdots \lambda_k x_k^2$, so the corresponding submatrix is not nilpotent. But since $F_i \in K[x_1, x_2, \dots, x_k]$ for all $i \le k$, the leading principal minor matrix of size k is nilpotent, because its p-th power is a submatrix of $(\mathcal{J}H)^p$. Contradiction.

- $t > n - 4 > 0$ *and for some* $i \le t$*, there does not exist a* $j \le t$ *such that* $H_j = \lambda_j x_i x_j^2$ *for some* $\lambda_j \in K^*$.
  Notice that $(\mathcal{J}H)|_{x_i=0}$ is nilpotent because $\mathcal{J}H$ is nilpotent. Since $F_i = x_i + H_i$ is reducible, it follows from 1) $\Longrightarrow$ 3) of corollary 4.4 that i-th row of $(\mathcal{J}H)|_{x_i=0}$ is zero. Hence the principal minor matrix that we obtain from $(\mathcal{J}H)|_{x_i=0}$ by removing its i-th row and i-th column is nilpotent as well. This minor matrix is equal to

$$\mathcal{J}_{x_1,\dots,x_{i-1},x_{i+1},\dots,x_n}\big(H_1|_{x_i=0}, \dots, H_{i-1}|_{x_i=0}, H_{i+1}|_{x_i=0}, \dots, H_n|_{x_i=0}\big)$$

  By 1) $\Longrightarrow$ 3) of corollary 4.4 we see that $x_j^2 \mid H_j$ for each $j \le t$. But by assumption on i, we have $x_i x_j^2 \nmid H_j$ for each $j \le t$. Hence by cubic homogeneity of H_j, we have $H_j|_{x_i=0} \ne 0$ for each $j \le t$ except $j = i$. So $\deg F_j = \deg F_j|_{x_i=0}$ and $\deg g = \deg g|_{x_i=0}$ for every $g \mid F_j$ and each $j \le t$ except $j = i$. As a consequence, $F_1|_{x_i=0}, \dots, F_{i-1}|_{x_i=0}, F_{i+1}|_{x_i=0}, \dots, F_t|_{x_i=0}$ are all reducible. Since $n-4$ is between 0 and t exclusive, we have $t \ge 2$, and it follows by induction on n that we get a contradiction.

So it remains to show that $t = s$. Suppose therefore that $t \ne s$. Then $t < s$ and by maximality of t, all linear combinations of $F_1, F_2, \dots, F_n$ which are reducible are already linear combinations of $F_1, F_2, \dots, F_t$. Since $t < s$, there exists a linear combination of the form $\mu_1 F_1 + \mu_2 F_2 + \cdots + \mu_t F_t$ which is reducible, such that $\mu_i \ne 0$ for at least two i's. Hence we may assume that there is a reducible

linear combination of the form $\mu_1 F_1 + \mu_2 F_2 + \cdots + \mu_r F_r$ with $\mu_1\mu_2\cdots\mu_r \ne 0$, where $2 \le r \le t$.

Assume first that there exist $i \le r < k$ such that $\frac{\partial}{\partial x_k} H_i \ne 0$. On account of 1) $\Longrightarrow$ 3) of corollary 4.4, for each $j \le r$, we have $H_j = x_j^2 g$ for some linear form g, which is the only irreducible factor of F_j that may contain x_k. Therefore,

$$\frac{\partial}{\partial x_k}(\mu_1 H_1 + \mu_2 H_2 + \cdots + \mu_r H_r) \tag{11}$$

is a nontrivial K-linear combination of $x_1^2, x_2^2, \dots, x_r^2$. Since the coefficient of x_1x_2 of $(\mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_r x_r)^2$ is $2\mu_1\mu_2 \ne 0$, $(\mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_r x_r)^2$ does not divide (11), and neither divides $\mu_1 H_1 + \mu_2 H_2 + \cdots + \mu_r H_r$. Now 1) $\Longrightarrow$ 3) of corollary 4.4 tells us that the Jacobian of $\mu_1 F_1 + \mu_2 F_2 + \cdots + \mu_r F_r$ is not unimodular. Hence $(\mu_1 F_1 + \mu_2 F_2 + \cdots + \mu_r F_r, F_2, F_3, \dots, F_n)$ is not a Keller map and neither is F. This contradicts that $\mathcal{J}H$ is nilpotent.

Assume next that $\frac{\partial}{\partial x_k} H_i = 0$ for all $i \le r < k$. Then $H_i \in K[x_1, x_2, \dots, x_r]$ for all $i \le r$ and the leading principal minor matrix of size r of $\mathcal{J}H$ is nilpotent because its p-th power is a submatrix of $(\mathcal{J}H)^p$. Since $r > r - 4$, it follows by induction on n that $\mathcal{J}_{x_1,x_2,\dots,x_r}(H_1, H_2, \dots, H_r)$ is not nilpotent. But the latter matrix is exactly the above principal minor matrix. Contradiction. □

Corollary 5.2. *Assume that $F \in K[x]^n$ is a cubic Keller map over a field K of characteristic zero without quadratic terms. Then there exists a $\lambda \in K^n$ such that for all $h \in K[x, x_{n+1}]$, the map*

$$G := (F - \lambda x_{n+1}^3, x_{n+1}, x_{n+2} - h)$$

has the property that

$$\mu_1 G_1 + \mu_2 G_2 + \cdots + \mu_{n+2} G_{n+2} + \mu_{n+3}$$

is irreducible for all $\mu \in K^{n+3}$ such that $\mu_i \ne 0$ for some $i \le n+2$.

Proof. Take T as in theorem 5.1 and let λ be the Hadamard sum of the columns of T. Assume that $g := \mu_1 G_1 + \mu_2 G_2 + \cdots + \mu_{n+2} G_{n+2} + \mu_{n+3}$ is reducible for some $\mu \in K^{n+3}$ such that $\mu_i \ne 0$ for some $i \le n+2$. If $f := \mu_1 F_1 + \mu_2 F_2 + \cdots + \mu_n F_n + \mu_{n+3} = g$, then $f(Tx) = g(Tx)$ is reducible as well, and by theorem 5.1, $(\mu_1\ \mu_2\ \cdots\ \mu_n)$ is c times a row of T^{-1} for some $c \in K^*$. By the choice of λ, we have $g = f + (\mu_1\ \mu_2\ \cdots\ \mu_n)\lambda x_{n+1}^3 = f + c x_{n+1}^3$ in this case, which contradicts $f = g$.

So $f \ne g$ and by lemma 4.7, $\mathcal{J}f$ is not unimodular and $\mu_{n+2} = 0$. Since $\mu_{n+1} G_{n+1} = \mu_{n+1} x_{n+1}$ is irreducible, we can choose $i \le n$. Hence $(F_1, \dots, F_{i-1}, f, F_{i+1}, \dots, F_n)$ is not a Keller map. This contradicts that F is a Keller map, so g is irreducible. □

6 Irreducibility lemmas for symmetric Keller maps

Lemma 6.1. *Assume F is a Keller map in dimension n over a field K of characteristic zero, such that $\mathcal{J}F$ is symmetric. If F_i is of the form $c' x_i + x_i^2 h + c$ for some $c, c' \in K$ and a $h \in K[x]$, then $h = 0$.*

Proof. Suppose that $h \neq 0$, say that $h = x_i^{r-2}\tilde{h}$, where $x_i \nmid \tilde{h}$. Then $F_i = c'x_i + x_i^r\tilde{h} + c$ and from Poincaré's lemma, it follows that $F = \nabla f$ for some $f \in K[x]$. Hence $\frac{\partial}{\partial x_i} f = F_i$, so if $\tilde{h}$ has no terms that are divisible by x_i, then

$$f = f|_{x_i=0} \cdot 1 + c \cdot x_i + \tfrac{1}{2}c' \cdot x_i^2 + \tfrac{1}{r+1}\tilde{h} \cdot x_i^{r+1}$$

In the general case, we can turn out terms of $\tilde{h}$ that are divisible by x_i by reducing $\tilde{h} \cdot x_i^{r+1}$ modulo x_i^{r+2}, and we have

$$f \bmod x_i^{r+2} = f|_{x_i=0} \cdot 1 + c \cdot x_i + \tfrac{1}{2}c' \cdot x_i^2 + \tfrac{1}{r+1}\tilde{h}|_{x_i=0} \cdot x_i^{r+1}$$

It follows that $\frac{\partial}{\partial x_i} F_i = \frac{\partial^2}{\partial x_i^2} f$ is the only entry of the matrix $\mathcal{J}F = \mathcal{H}f$ with terms of degree between 1 and $r-1$ inclusive in x_i, and those terms add up to $r\tilde{h}|_{x_i=0} \cdot x_i^{r-1} \neq 0$, because $r \geq 2$ by definition. If t is a term of $\tilde{h}|_{x_i=0}$ with nonzero coefficient, say with coefficient $c'' \neq 0$, then the coefficients of 1 and $x_i^{r-1}t$ of the i-th row $\mathcal{J}F_i$ of $\mathcal{H}f$ are the row vectors

$$(0^1 \ \cdots \ 0^{i-1} \ c' \ 0^{i+1} \ \cdots \ 0^n) \qquad \text{and} \qquad (0^1 \ \cdots \ 0^{i-1} \ rc'' \ 0^{i+1} \ \cdots \ 0^n) \tag{12}$$

respectively. Since $\det \mathcal{J}F \in K^*$, the constant part $\det(\mathcal{J}F)|_{x=0}$ of $\det \mathcal{J}F$ is nonzero, and $c' \neq 0$ follows.

Suppose now that the term t of $\tilde{h}|_{x_i=0}$ is not divisible by any other such term with nonzero coefficient. Then the term $rc''x_i^{r-1}t$ of $(\mathcal{J}F)_{ii}$ is the only nonzero term in $\mathcal{J}F$ that both divides $x_i^{r-1}t$ and is divisible by x_i. Hence the coefficient of $x_i^{r-1}t$ in $\det \mathcal{J}F = \det \mathcal{H}f$ is formed by the term $rc''x_i^{r-1}t$ of $(\mathcal{J}F)_{ii}$ and other terms of $\mathcal{J}F$, which have to be constant terms, being outside row i of $\mathcal{J}F$ because of (12).

(12) additionally tells us that inside row i of $\mathcal{J}F$, the coefficients of $x_i^{r-1}t$ are a factor $rc''(c')^{-1}$ larger than those of 1. Hence by linearity of the determinant function in the i-th row, the coefficient of $x_i^{r-1}t$ in $\det \mathcal{J}F = \det \mathcal{H}f$ is equal to $rc''(c')^{-1}$ times the constant part $\det(\mathcal{J}F)|_{x=0}$ of $\det \mathcal{J}F$. This contradicts $\det \mathcal{J}F \in K^*$, so $h = 0$. □

Corollary 6.2. *Assume G is a Keller map in dimension n over a field K of characteristic zero, such that $\mathcal{J}G$ is symmetric. Write $G_i^{(1)}$ for the linear part of G_i and assume that $\frac{\partial}{\partial x_i} G_i^{(1)} \neq 0$.*

If G_i is of the form $G_i^{(1)} + \big(G_i^{(1)}\big)^2 h + c$ for some $c \in K$ and a $h \in K[x]$, then $h = 0$ and hence $\deg G_i = 1$.

Proof. Take $T \in \mathrm{GL}_n(K)$ such that T corresponds to the identity matrix I_n except for the i-th row, for which we take $\big(\frac{\partial}{\partial x_1} G_i^{(1)} \ \frac{\partial}{\partial x_2} G_i^{(1)} \ \cdots \ \frac{\partial}{\partial x_n} G_i^{(1)}\big)$. From Poincaré's lemma, it follows that $G = \nabla g$ for some $g \in K[x]$. Next, define $f := g(T^{-1}x)$ and $F := (T^{-1})^{\mathrm{t}} G(T^{-1}x)$. Since $(T^{-1})^{\mathrm{t}}$ corresponds to the identity matrix I_n except for the i-th column, we have $F_i = ((T^{-1})^{\mathrm{t}})_i G(T^{-1}x) = (T^{-1})_{ii} G_i(T^{-1}x)$.

By definition of T, the linear part of G_i is equal to Tx, thus the linear part of $G_i(T^{-1}x)$ is equal to $G_i^{(1)}(T^{-1}x) = x_i$. Hence $F_i = (T^{-1})_{ii} G_i(T^{-1}x) = (T^{-1})_{ii}\big(x_i + x_i^2 h(T^{-1}x) + c\big)$. Furthermore,

$$\nabla f = (\mathcal{J} f)^{\mathrm{t}} = \big(\mathcal{J} g(T^{-1}x)\big)^{\mathrm{t}} = \big(G^{\mathrm{t}}|_{x=T^{-1}x} \cdot T^{-1}\big)^{\mathrm{t}} = (T^{-1})^{\mathrm{t}} G(T^{-1}x) = F$$

so $\mathcal{J} F$ is symmetric, and we have $h(T^{-1}x) = 0$ on account of lemma 6.1. This gives the desired result. □

Theorem 6.3. *Assume G is a Keller map in dimension n over a field K of characteristic zero, such that $\mathcal{J} G$ is symmetric. Write $G_i^{(k)}$ for the homogeneous part of degree k of G_i and suppose that $\frac{\partial}{\partial x_i} G_i^{(1)} \neq 0$ and $G_i^{(1)} \mid G_i^{(k)}$ for all $k < d$.*

If G_i has a divisor of degree less than d with trivial constant part, then $\deg G_i = 1$.

Proof. Suppose that $g \mid G_i$, $\deg g < d$ and $g(0) = 0$. Say that $G_i = gh$. Since $g(0) = 0$, we have $h(0) \mid G_i^{(1)} \neq 0$. From lemma 4.2 with $g^* = G_i^{(1)}$, we obtain that $G_i^{(1)} \mid g \mid G_i$. Hence we may assume that $g = G_i^{(1)}$.

By the Keller condition, G_i is nonsingular over the algebraic closure of K, which gives that G_i is of the form of corollary 6.2 above with $c = 0$. On account of that corollary, $\deg G_i = 1$. □

Lemma 6.4. *Let f be of the form $g_0 + g_1 y_1 + g_2 y_2 + \cdots + g_n y_n$, where $g_i \in A$ for all i for some unique factorization domain A. Then f is irreducible, if and only if $\gcd\{g_0, g_1, g_2, \ldots, g_n\} \in A^*$.*

Proof. If f decomposes in two factors, then one of this factors is constant with respect to y. The rest of the proof is an easy exercise. □

Lemma 6.5. *Assume that $F \in K[x]^n$ is a polynomial map over a field K of characteristic zero and $f \in K[x]$. Set $G := \nabla_{x,y}(f + y^{\mathrm{t}} F)$. Then $\det \mathcal{J}_{x,y} G = (-1)^n (\det \mathcal{J} F)^2$ and F is invertible, if and only if G is invertible. Furthermore,*

$$\mu_1 G_1 + \mu_2 G_2 + \cdots + \mu_n G_n + \cdots + \mu_{2n} G_{2n} + \mu_{2n+1} \tag{13}$$

is irreducible for all $\mu \in K^{2n+1}$ such that $\mu_i \neq 0$ for some $i \leq n$ in case F is a Keller map.

Proof. Notice that $\mathcal{J}_{x,y} G$ is of the form

$$\mathcal{J}_{x,y} G = \mathcal{H}_{x,y}(f + y^{\mathrm{t}} F) = \begin{pmatrix} * & (\mathcal{J} F)^{\mathrm{t}} \\ (\mathcal{J} F) & \emptyset \end{pmatrix}$$

whence $\det \mathcal{J}_{x,y} G = (-1)^n (\det \mathcal{J} F)^2$. Consequently, both F and G are Keller maps in case one of them is, which we assume from now on. Notice that $G = \big((\mathcal{J} F)^{\mathrm{t}} y + \nabla f, F\big)$ by definition. Hence we have

$$G\Big(x, \big((\mathcal{J} F)^{\mathrm{t}}\big)^{-1}(y - \nabla f)\Big) = (y, F)$$

and we see that that G is invertible, if and only if F is invertible.

Suppose that $\mu \in K^{2n+1}$ such that $\mu_i \neq 0$ for some $i \leq n$. Then (13) is of the form $g_0 + g_1 y_1 + g_2 y_2 + \cdots + g_n y_n$ with $g_i \in K[x]$ for all i. More precisely,

$$(g_1 \; g_2 \; \cdots \; g_n) = \mathcal{J}_y \Big((\mu_1 \; \mu_2 \; \cdots \; \mu_n) \big((\mathcal{J}F)^{\mathrm{t}} y + \nabla f \big) \Big)$$

is a nontrivial linear combination of the rows of $(\mathcal{J}F)^{\mathrm{t}}$, so $(g_1 \; g_2 \; \cdots \; g_n)$ is unimodular. In particular, $\gcd\{g_0, g_1, g_2, \ldots, g_n\} \in K^*$. Hence (13) is irreducible on account of lemma 6.4. □

References

[Bak1] Sławomir Bakalarski. Jacobian problem for factorial varieties. *Univ. Iagel. Acta Math.*, (44):31–34, 2006.

[Bak2] Sławomir Bakalarski. The irreducibility of symmetric Yagzhev maps. *Proc. Amer. Math. Soc.*, 138(7):2279–2281, 2010.

[BCW] Hyman Bass, Edwin H. Connell, and David Wright. The Jacobian conjecture: reduction of degree and formal expansion of the inverse. *Bull. Amer. Math. Soc. (N.S.)*, 7(2):287–330, 1982.

[dBvdE1] Michiel de Bondt and Arno van den Essen. The Jacobian conjecture for symmetric Drużkowski mappings. *Ann. Polon. Math.*, 86(1):43–46, 2005.

[dBvdE2] Michiel de Bondt and Arno van den Essen. A reduction of the Jacobian conjecture to the symmetric case. *Proc. Amer. Math. Soc.*, 133(8):2201–2205 (electronic), 2005.

[CvdE] Charles Ching-An Cheng and Arno van den Essen. Endomorphisms of the plane sending linear coordinates to coordinates. *Proc. Amer. Math. Soc.*, 128(7):1911–1915, 2000.

[Dru1] Ludwik M. Drużkowski. An effective approach to Keller's Jacobian conjecture. *Math. Ann.*, 264(3):303–313, 1983.

[Dru2] Ludwik M. Drużkowski. The Jacobian conjecture: symmetric reduction and solution in the symmetric cubic linear case. *Ann. Polon. Math.*, 87:83–92, 2005.

[vdE] Arno van den Essen. *Polynomial automorphisms and the Jacobian conjecture*, volume 190 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2000.

[vdES] Arno van den Essen and Vladimir Shpilrain. Some combinatorial questions about polynomial mappings. *J. Pure Appl. Algebra*, 119(1):47–52, 1997.

[Hub] E.-M.G.M. Hubbers. The Jacobian Conjecture: Cubic homogeneous maps in dimension four. Master's thesis, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands, February 17 1994. Directed by A.R.P. van den Essen.

[Jag] A. V. Jagžev. On a problem of O.-H. Keller. *Sibirsk. Mat. Zh.*, 21(5):141–150, 191, 1980.

[Jęd] Piotr Jędrzejewicz. A characterization of Keller maps. *J. Pure Appl. Algebra*, 217(1):165–171, 2013.

[Jel1] Zbigniew Jelonek. A solution of the problem of van den Essen and Shpilrain. *J. Pure Appl. Algebra*, 137(1):49–55, 1999.

[Jel2] Zbigniew Jelonek. A solution of the problem of van den Essen and Shpilrain II. *J. Algebra*, 358:8–15, 2012.

[Kal] Shulim Kaliman. On the Jacobian conjecture. *Proc. Amer. Math. Soc.*, 117(1):45–51, 1993.

[Kel] Ott-Heinrich Keller. Ganze Cremona-Transformationen. *Monatsh. Math. Phys.*, 47(1):299–306, 1939.

[KS] Tadeusz Krasiński and Stanisław Spodzieja. On the irreducibility of fibres of complex polynomial mappings. *Univ. Iagel. Acta Math.*, (39):167–178, 2001. Effective methods in algebraic and analytic geometry, 2000 (Kraków).

[LM] Jeffrey Lang and Samer Maslamani. Some results on the Jacobian conjecture in higher dimension. *J. Pure Appl. Algebra*, 94(3):327–330, 1994.

[MSY] Alexander A. Mikhalev, Vladimir Shpilrain, and Jie-Tai Yu. On endomorphisms of free algebras. *Algebra Colloq.*, 6(3):241–248, 1999.

[Sch] A. Schinzel. *Polynomials with special regard to reducibility*, volume 77 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2000. With an appendix by Umberto Zannier.

[Wan] Stuart Sui Sheng Wang. A Jacobian criterion for separability. *J. Algebra*, 65(2):453–494, 1980.